

Phase transitions due to interaction between photons and atoms in a cavity system

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We survey cooperative phenomena of a cavity system in which many atoms interact with a single quantized photon mode driven by an AC external field in a dissipative environment. It has been known that an external field causes the optical bistability which is a non-equilibrium phase transition for the balance of excitation and dissipation. On the other hand, a strong interaction causes the Dicke transition which takes place with a spontaneous appearance of photons and excited atoms in the equilibrium system. We study the phenomena in full range of the strengths of the interaction and the external field. When the interaction is strong, the simple Lindblad equation cannot reproduce the correct equilibrium state. In order to treat the highly interacting region, we provide an extended master equation which realizes the ground state in the case of no external field. We study the Dicke model and the Tavis-Cummings model under an AC external field by using the master equation and obtain the two following results: first, we present phase diagrams of the stationary state of the two models and we find that the rotating wave approximation causes a qualitative different nature of the phase diagrams when the interaction and the external field are strong. Second, we find a novel symmetry breaking phenomenon induced by the AC external field in the Dicke model.

I. INTRODUCTION

The effect of the interaction between photons and atoms has been studied for a long time. A cavity confines photons in a finite region with mirrors, which enhances the interaction between photons and atoms. When a single atom is regarded to be confined in a cavity, the effect of interaction has been studied by using the Jaynes-Cummings model [1, 2], where a single quantized mode of the cavity photons couples with a two-level system. For the system with many atoms, the extension of the Jaynes-Cummings model has been introduced as the Tavis-Cummings model [3]. With increasing the number of atoms, the hybridization of the two-level systems and photon system is enhanced [4]. In experiments, we observe the effect of interaction as cavity ringing phenomena [5] and the vacuum field Rabi splitting [6, 7]. Recently, this coupling leads to the ultra-strong coupling regime [8, 9], and also attracts much attention as a possible method to control to exchange quantum information between two-level systems and photon system [10–16].

We also observe cooperative phenomena due to the interaction in a cavity. Even if there is no direct interaction between atoms, an effective long-range interaction appears among atoms via a common mode of cavity photons. It has been pointed out that the system shows two kinds of phase transitions: the Dicke transition and the optical bistability. The former is regarded as a ground-state phase transition as a function of the coupling strength at zero temperature [17, 18]. When the coupling strength is weak, the ground state of the system is a state without excitations of photons and atoms. When the interaction exceeds a critical value, photons and dipole moments of atoms appear spontaneously as a continuous transition. The latter is a kind of non-equilibrium phase transition under AC external field in contact with thermal baths. It has been found that the stationary state changes discontinuously as a function of the strength and the frequency of the external field. This is called the optical bistability [19–21]. Recently, a phase transition has been also predicted when the strength of interaction is driven periodically [22]. Furthermore, thanks to the recent realization of the atom-photon coupling in various kinds of experiments, many cooperative phenomena are studied extensively from experimental and theoretical views [23].

In the present paper, we study cooperative phenomena in a cavity due to the strong interaction and strong external field in a dissipative environment. As a model of a cavity, we adopt the Tavis-Cummings model [3] and the Dicke model [24], in which many two-level systems couple with a single quantized mode of the cavity photons. In order to

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describe the system under a dissipative environment, we use a quantum master equation. In the literatures [25, 26], the simple Lindblad equation is adopted. In those cases, the effect of the atom-photon coupling does not reflect on the dissipative term. Such a treatment is enough to study the system with relatively weak couplings. Indeed the optical bistability has been well studied with the simple Lindblad equation [25, 26]. However, within this scheme, we cannot reproduce the ground state correctly as a stationary state in the case of no external field. In order to study overall structure of the phase diagram parameterized by the strength of the interaction and the intensity of the external field, we construct an extended master equation in which the effect of the interaction is taken into account. With this master equation, we reproduce the ground state correctly as a stationary state of the master equation.

We obtain a stationary state by using the master equation and classify the stationary states into three phases by taking into account the symmetry of the time-evolution operators. The three phases are analyzed by the Fourier spectrum of the long-time behavior. We find the following three phases: “regularly oscillating phase” when the system is oscillating with the external field, “ordered phase” when the system has a spontaneous order, and “irregularly oscillating phase” when the system shows an irregular oscillation.

We obtain two results: first, we obtain the phase diagram of the Tavis-Cummings model and the Dicke model under the AC external field. We find that the two models show a qualitatively different behavior when the interaction and the AC external field are strong and that the rotating wave approximation is not appropriate in this region. Second, we find a novel symmetry breaking phenomenon which is induced by the AC external field in the case of the Dicke model.

The paper consists of the following sections. In Sec. II, the model of the system is explained. In Sec. III, a master equation which takes into account the atom-photon coupling is derived. In Sec. IV, we derive the equations of motions with mean-field (MF) treatment, and we show that ground states are realized in our approach in the case of no external field. In Sec. V, we study the phase diagram in the case of the Tavis-Cummings model. In Sec. VI, we study the phase diagram in the case of the Dicke model. In Sec. VII, the summary and discussion are given.

II. SYSTEM HAMILTONIAN

We study the system of photons and atoms in a cavity with an external field given by

$$\frac{\mathcal{H}_S(t)}{\hbar} = \omega_p a^\dagger a + \sum_{i=1}^N \omega_a S_i^z + \tilde{g} \sum_{i=1}^N \{ (aS_i^+ + a^\dagger S_i^-) + \chi (a^\dagger S_i^+ + a S_i^-) \} + 2\tilde{\xi} (a^\dagger + a) \cos(\omega_e t). \quad (1)$$

The first term represents the single quantized mode of the cavity photons with the resonance frequency ω_p which depends on the form of the cavity. The annihilation and creation operators for the cavity photon are denoted by a and a^\dagger , respectively. These operators satisfy the bosonic commutation relation $[a, a^\dagger] = 1$. In the second term, we describe many atoms as an ensemble of two-level systems with the energy gap ω_a which are described by spin one-half operators. The spin operator $\mathbf{S}_i (i = 1, \dots, N)$ represents the i th two-level system with the commutation relation: $[S_i^+, S_i^-] = 2S_i^z$. The third term gives the interaction between photons and atoms, and \tilde{g} is its coupling constant between them. The last term describes the AC external field with the frequency ω_e and the amplitude $\tilde{\xi}$.

We study the time evolutions of the following quantities: photon field in the cavity $\langle a \rangle$, the number of excited atoms which is described by the z -component of the total magnetization $\langle \sum_{i=1}^N S_i^z \rangle$, and the dipole moments of atoms which are described by the transverse-component of spin operator $\langle \sum_{i=1}^N S_i^+ \rangle$. In the present study, we study ξ -dependence for the case of $\omega_a = \omega_p = \omega_e$, though ω_e dependence on photon emission is an interesting problem and has been studied extensively in experiment [21].

The Tavis-Cummings model is given by applying the rotating wave approximation to the Dicke model, which corresponds to the case $\chi = 0$ while the Dicke model is defined by the case $\chi = 1$. In the two models, the nature of symmetry is different. the Tavis-Cummings model has $U(1)$ symmetry where the ordered components $\langle a \rangle = |\langle a \rangle| e^{i\phi}$ and $\sum_{i=1}^N \langle S_i^+ \rangle / N = -|\sum_{i=1}^N \langle S_i^+ \rangle / N| e^{-i\phi}$ can have any phase ϕ . On the other hand, The Dicke model has Z_2 symmetry where the ordered components are given by $\langle a + a^\dagger \rangle$ and $\sum_{i=1}^N \langle S_i^+ + S_i^- \rangle / N$. This difference is important when we study the symmetry breaking phenomena under the AC external field as we will see in Secs. VI and VII.

In the present work, we study the macroscopic properties in the thermodynamic limit: $N \rightarrow \infty$ and $V \rightarrow \infty$ keeping N/V to be a constant. Here, V is a volume of the cavity. In this case, we expect that the expectation values of $a^\dagger a$, $\sum_{i=1}^N S_i^z$, and $\sum_{i=1}^N S_i^\pm$ are of $O(N)$. Consequently, the expectation value of the term $\sum_{i=1}^N (aS_i^+ + a^\dagger S_i^-) + \chi(a^\dagger S_i^+ + a S_i^-)$ is of $O(N\sqrt{N})$. To obtain the meaningful result in the thermodynamic limit, we introduce scaled parameters.

First, we introduce a scaled interaction between photons and atoms as

$$g = \sqrt{N} \tilde{g}. \quad (2)$$

The coupling constant \tilde{g} in Hamiltonian (1) is given by

$$\tilde{g} = \left| \frac{\omega_a}{\sqrt{2\epsilon_0\omega_p V}} \langle \text{down} | e \sum_{j=1}^{N_e} \mathbf{x}_j^{(i)} \cdot \mathbf{e} | \text{up} \rangle \right|, \quad (3)$$

where $\mathbf{x}_j^{(i)}$ denotes the position of the j th electron in the i th atom, and $|\text{down}\rangle$ and $|\text{up}\rangle$ express the eigenstates of $S_i^z = -1/2$ and $S_i^z = 1/2$, respectively. We assume that the value of \tilde{g} does not depend on atoms. Here, e is the elementary charge and \mathbf{e} is the polarization vector of the cavity photon. Because \tilde{g} is inversely proportional to the square root of the volume of the cavity V , the scaled value g is of $O(1)$ in the thermodynamic limit. With this scaling, the interaction term $\tilde{g}(a + a^\dagger) \sum_{i=1}^N (S_i^+ + S_i^-)$ scales as $O(N)$.

Next, we also rescale the external field as

$$\xi = \frac{\tilde{\xi}}{\sqrt{N}}. \quad (4)$$

Here, we assume that the amplitude of the external field increases proportional to \sqrt{N} . Then, the Hamiltonian is expressed with the new parameters as

$$\frac{\mathcal{H}_S(t)}{\hbar} = \omega_p a^\dagger a + \sum_{i=1}^N \omega_a S_i^z + \frac{g}{\sqrt{N}} \sum_{i=1}^N [(a S_i^+ + a^\dagger S_i^-) + \chi (a^\dagger S_i^+ + a S_i^-)] + 2\sqrt{N}\xi (a^\dagger + a) \cos(\omega_e t). \quad (5)$$

In the present paper, we study the nature of this Hamiltonian in the thermodynamic limit.

III. MASTER EQUATION

A. Stationary state in dissipative environment

We investigate the time evolution of the system in a dissipative environment. The dissipation comes from the interactions between the system and the thermal baths. The master equation has the form

$$\frac{\partial \rho_S}{\partial t} = -\frac{i}{\hbar} [\mathcal{H}_S(t), \rho_S] + \Gamma[\rho_S], \quad (6)$$

where $\Gamma[\rho_S]$ is a linear operator on ρ_S for the relaxation process.

After a long time evolution, the system reaches to a kind of stationary state at which the external field and the dissipation are balanced. In literatures [25, 26], the following simple Lindblad equation

$$\Gamma[\rho_S] = -\kappa ([a^\dagger, a\rho_S] + [\rho_S a^\dagger, a]) - \gamma_{xy} \sum_{i=1}^N ([S_i^+, S_i^- \rho_S] + [\rho_S S_i^+, S_i^-]) - \gamma_z \sum_{i=1}^N ([S_i^z, S_i^z \rho_S] + [\rho_S S_i^z, S_i^z]), \quad (7)$$

was adopted. In this case $\Gamma[\rho_S]$ represents the photon damping, the longitudinal damping of atoms $\gamma_{//} \equiv (\gamma_{xy} + 2\gamma_z)$, and the phase dephasing $\gamma_\perp \equiv 2\gamma_{xy}$. In this form, the effect of the atom-photon coupling in the system Hamiltonian does not reflect on the damping process. This effect was taken into account in some studies for the case of a single atom interacting with a cavity mode [27, 28], while it has not been considered in a multiatom case. In the multiatom case, it is more important to take into account this effect because this effect changes the nature of phases as we will see in the present paper.

In the present model (5), it is known that when g is $O(\sqrt{\omega_a \omega_p})$, the Dicke transition occurs in equilibrium ($\xi = 0$) by changing the coupling strength g [17, 18]. When g is greater than a critical value $O(\sqrt{\omega_a \omega_p})$, the photon field and dipole moments of atoms appear spontaneously. This ordered region is called “superradiant phase”. Because the relaxation terms in Eq. (7) always cause the reduction of photons and the relaxation of excited atoms, the master equation cannot correctly reproduce the ordered phase as a stationary state as we will see in Sec. IV.C. In the case of the Tavis-Cummings model, this simple Lindblad equation fails to have an ordered state at all. In the case of the Dicke model, it gives a phase transition, but fails to give the correct values of the order parameter. That is why it is necessary to obtain the master equation by projecting out the degrees of the freedom of the thermal baths, which incorporates effect of the atom-photon coupling into the relaxation terms.

B. General scheme for the master equation

We have derived the master equation by tracing out the degrees of the thermal baths. For simplicity, we consider the thermal baths at zero temperature for photons and atoms, respectively. As for the coupling between atoms and a dissipative environment, we may adopt several types of interactions. In reality, because the range of the dissipative environment is spatially, a uniform coupling with all atoms takes place. We consider a thermal bath which models this effect, and call it “global-coupling bath”. When there only exists a global-coupling bath, the system conserves the total spin, $\mathbf{S}^2 = \sum_{i=1}^N (S_i^+ S_i^- + S_i^- S_i^+)/2 + (S_i^z)^2$. On the other hand, if we take into account the thermal bath coupling with each atom independently, the total spin is no longer conserved. In this case, we call the bath “local-coupling bath”. In order to study the relaxation phenomena, we need to take into account the local-coupling bath. In principle we have to contain both of the global and local coupling baths. In Appendix. C, we study dependence of the stationary states on the type of dissipative environment. If we include only the global-coupling bath, the system has an additional conserved quantity and the qualitative nature of the stationary states changes. In contrast, we find that the qualitative nature of the stationary states is similar as long as a contribution of the local-coupling bath is included. Therefore, in the present paper we adopt only the local-coupling bath as in previous works [25, 26], which represents the latter cases.

Each thermal bath consists of free bosons and the Hamiltonian of thermal baths is given by

$$\mathcal{H}_B = \sum_{n=0}^N \sum_{\omega_\alpha(>0)} \hbar \omega_\alpha A_n^\dagger(\omega_\alpha) A_n(\omega_\alpha), \quad (8)$$

where $A_n(\omega_\alpha)$ is the boson operator of each bath. We use the subscript $n = 0$ for the photon and $n = 1, \dots, N$ for the n th atom. The interaction Hamiltonian is given by

$$\mathcal{H}_I = \hbar \lambda \sum_{n=0}^N X_n Y_n, \quad (9)$$

where

$$X_0 = a + a^\dagger, \quad X_i = S_i^- + S_i^+ \quad (i = 1, 2, \dots, N), \quad (10)$$

$$Y_n = \sum_{\omega_\alpha(>0)} k_n(\omega_\alpha) A_n^\dagger(\omega_\alpha) + k_n^*(\omega_\alpha) A_n(\omega_\alpha), \quad (11)$$

and λ is a coefficient to describe the strength of the coupling. In the present study, for simplicity, we consider the situation where the relaxation is only due to a radiative process, and thus we do not take into account the dissipation through S_i^z . The quantum master equation of the system is derived in the standard way [29–31]. As a result, we obtain

$$\begin{aligned} \frac{\partial \rho_S(t)}{\partial t} &= -\frac{i}{\hbar} [\mathcal{H}_S(t), \rho_S(t)] \\ &\quad - \lambda^2 \sum_{n=0}^N \int_0^\infty du \int_0^\infty d\omega \left| \tilde{k}_n(\omega) \right|^2 \\ &\quad \times \left(e^{-i\omega u} [X_n, X_n(t, t-u) \rho_S(t)] - e^{i\omega u} [X_n, \rho_S(t) X_n(t, t-u)] \right). \end{aligned} \quad (12)$$

where

$$X_n(t, \tau) = \text{T exp} \left(\int_\tau^t d\tau' \frac{i}{\hbar} \mathcal{L}_S(\tau') \right) X_n, \quad (13)$$

$$\left| \tilde{k}_n(\omega) \right|^2 = \sum_{\omega_\alpha(>0)} |k_n(\omega_\alpha)|^2 \delta_{\omega_\alpha, \omega} \quad (14)$$

Here, we define the Liouville operator as $i\mathcal{L}_S(t) \cdot \equiv -i[\mathcal{H}_S(t), \cdot]$.

In the derivation, we consider the Van Hove limit (weak-coupling limit) [32] where $\lambda \rightarrow 0$ keeping $t\lambda^2$ to be constant. In this limit, the relaxation time of the system into the stationary state $O(1/\lambda^2)$ is much larger than the typical

correlation time between the system and the thermal bath, and it is justified to neglect the contribution from the higher terms of \mathcal{H}_I than the second order and to apply the Markov approximation [33]. We also ignore the term which represents the influence of initial correlations between the system and baths, which is shown in [34] in the Van Hove limit to give the same result as one which a general initial state gives.

In addition, in order to approximate the master equation into tractable form, we consider the case when correlation time is much smaller than the period of the external field. In this case, we may write

$$X_n(t, t-u) \simeq \exp\left(-\frac{i}{\hbar}\mathcal{H}_S(t)u\right) X_n \exp\left(\frac{i}{\hbar}\mathcal{H}_S(t)u\right). \quad (15)$$

Here we regard the external field as a constant during the time evolution from $t-u$ to t . With this approximation, the master equation is given by

$$\begin{aligned} \frac{\partial \rho_S(t)}{\partial t} &= \frac{1}{i\hbar} [\mathcal{H}_S(t), \rho_S(t)] \\ &\quad - \lambda^2 \sum_{n=0}^N \int_0^\infty du \int_0^\infty d\omega \left| \tilde{k}_n(\omega) \right|^2 \\ &\quad \times \left\{ e^{-i\omega u} \left[X_n, \exp\left(-\frac{i}{\hbar}\mathcal{H}_S(t)u\right) X_n \exp\left(\frac{i}{\hbar}\mathcal{H}_S(t)u\right) \rho_S(t) \right] \right. \\ &\quad \left. - e^{i\omega u} \left[X_n, \rho_S(t) \exp\left(-\frac{i}{\hbar}\mathcal{H}_S(t)u\right) X_n \exp\left(\frac{i}{\hbar}\mathcal{H}_S(t)u\right) \right] \right\}, \\ &\equiv i\mathcal{L}_T(t) \rho_S. \end{aligned} \quad (16)$$

It is noted that the master equation is applicable regardless of the number of atoms and the scaling of the coupling constant. If we ignore effects of the interaction and the external field in the Hamiltonian, we arrive at the form (7) (Appendix A).

C. The stationary state and the averaged values of physical quantities

In the present paper we study dynamics evolved by Eq. (16). Because the system is driven by the external field, the meaning of the stationary state is obscure. However, when we consider the symmetry of the time evolution operator $\mathcal{L}_T(t)$, it is possible to define the stationary states and obtain the relation on the averaged values over the period of the external field.

For the definition of the stationary state, we may use the Floquet eigenstates because the time evolution operator $i\mathcal{L}_T(t)$ is periodic in time,

$$i\mathcal{L}_T(t + T_e) = i\mathcal{L}_T(t) \quad \left(T_e \equiv \frac{2\pi}{\omega_e}\right). \quad (17)$$

The eigenstate of Floquet operator \mathcal{F} is given by

$$\mathcal{F}\phi_F(0) \equiv \text{T exp}\left(i \int_0^{T_e} dt' \mathcal{L}_T(t')\right) \phi_F(0) \equiv e^{-i\epsilon_F T_e} \phi_F(0), \quad (18)$$

where $\phi_F(0)$ and ϵ_F are the Floquet eigenfunction and quasi-eigenenergy, respectively. We expand $\rho_S(0)$ with the Floquet eigenfunctions.

$$\rho_S(0) = \sum_F c_F \phi_F(0), \quad (19)$$

where c_F is a coefficient. We define the time evolution of the Floquet eigenfunction $\phi_F(t)$ as

$$\phi_F(t) \equiv \text{T exp}\left(i \int_0^t \mathcal{L}_T(t') dt'\right) \phi_F(0). \quad (20)$$

In this definition, for all t ,

$$\rho_S(t) = \sum_F c_F \phi_F(t), \quad (21)$$

and

$$\phi_F(t + T_e) = e^{-i\epsilon_F T_e} \phi_F(t). \quad (22)$$

If the imaginary part of ϵ_F is zero, the corresponding state $\phi_F(t)$ remains in the long-time limit. We therefore define the stationary state $\rho_{S,SS}(t)$,

$$\rho_{S,SS}(t) = \sum_{F \in S} c_F \phi_F(t), \quad S = \{F \mid \text{Im} \epsilon_F = 0\}, \quad (23)$$

where S is a set of eigenfunctions with $\text{Im} \epsilon_F = 0$.

To discuss the relation on averaged physical values over the period of the external field, we introduce the unitary operator

$$U = \exp \left[i\pi \left(a^\dagger a + \sum_{i=1}^N S_i^z \right) \right], \quad (24)$$

which changes the sign of the operators a, a^\dagger, S_i^+ , and S_i^- such that $U^\dagger a U = -a$. Under this unitary transformation, we have the relation

$$U^\dagger \mathcal{L}_T(t) U = \mathcal{L}_T \left(t + \frac{T_e}{2} \right), \quad (25)$$

because of the following facts. In the Hamiltonian (5), all the terms except the last one are invariant with the transformation of U , while $a + a^\dagger$ of the last term is changed to $-(a + a^\dagger)$. Under time translation $t \rightarrow t + T_e/2$, $\cos(\omega_e t)$ changes the sign and the last term is also invariant. Thus, Eq. (25) holds. With this property, we can show that $U \phi_F(T_e/2) U^\dagger$ is an eigenfunction of \mathcal{F} with the same quasi-eigenenergy of $\phi_F(0)$: Because of Eq. (18),

$$\begin{aligned} \mathcal{F} U \phi_F \left(\frac{T_e}{2} \right) U^\dagger &= U \phi_F \left(\frac{3T_e}{2} \right) U^\dagger \\ &= e^{-i\epsilon_F T_e} U \phi_F \left(\frac{T_e}{2} \right) U^\dagger. \end{aligned} \quad (26)$$

Now we consider averaged values $\langle f(a, a^\dagger, S_i^\pm, S_i^z) \rangle_t$ in the stationary states. Here $\langle \dots \rangle_t$ means $\text{Tr} \dots \rho_{S,SS}(t)$, and f is an arbitrary function of operators.

If the stationary state is unique, the set S consists of only one element F_0 and the stationary state is given by

$$\rho_{S,SS}(t) = \phi_{F_0}(t). \quad (27)$$

In this case, ϵ_{F_0} must be zero because $\text{Tr} \rho_{S,SS}(t) = 1$ must be satisfied. We find from Eq. (26) and the uniqueness of the stationary state that

$$\phi_{F_0}(0) = U \phi_{F_0} \left(\frac{T_e}{2} \right) U^\dagger. \quad (28)$$

In this case, we conclude

$$\begin{aligned} \langle f(a, a^\dagger, S_i^\pm, S_i^z) \rangle_t &= \text{Tr} f(a, a^\dagger, S_i^\pm, S_i^z) \rho_{S,SS}(t) \\ &= \text{Tr} f(a, a^\dagger, S_i^\pm, S_i^z) \text{T exp} \left(i \int_0^t dt' \mathcal{L}_T(t') \right) \phi_{F_0}(0) \\ &= \text{Tr} f(a, a^\dagger, S_i^\pm, S_i^z) \text{T exp} \left(i \int_0^t dt' \mathcal{L}_T(t') \right) U \phi_{F_0} \left(\frac{T_e}{2} \right) U^\dagger \\ &= \text{Tr} U f(-a, -a^\dagger, -S_i^\pm, S_i^z) \text{T exp} \left(i \int_{T_e/2}^{t+T_e/2} dt' \mathcal{L}_T(t') \right) \phi_{F_0} \left(\frac{T_e}{2} \right) U^\dagger \\ &= \text{Tr} f(-a, -a^\dagger, -S_i^\pm, S_i^z) \rho_{S,SS} \left(t + \frac{T_e}{2} \right) \\ &= \langle f(-a, -a^\dagger, -S_i^\pm, S_i^z) \rangle_{t+T_e/2}. \end{aligned} \quad (29)$$

Averaging over the period of the external field, we have the relation

$$\overline{\langle f(a, a^\dagger, S_i^\pm, S_i^z) \rangle_t} = \overline{\langle f(-a, -a^\dagger, -S_i^\pm, S_i^z) \rangle_t}, \quad (30)$$

where $\overline{\cdots} = \int_0^{T_e} \cdots dt' / T_e$. For example, in the case that $f(a, a^\dagger, S_i^\pm, S_i^z) = a$,

$$\overline{\langle a \rangle_t} = \overline{\langle -a \rangle_t} = -\overline{\langle a \rangle_t}, \quad (31)$$

and thus

$$\overline{\langle a \rangle_t} = 0. \quad (32)$$

In a similar way, we can show

$$\overline{\langle S^+ \rangle_t} = 0. \quad (33)$$

Thus we find that averaged values of odd moments are zero if the stationary state is unique.

Depending on the structure of the set S, there are several types of stationary states. In the case that the set S consists of several elements, the stationary state is given by a linear combination of those modes and it looks irregular. In the thermodynamic limit, the system may exhibit a phase transition. In this case, the set of S is degenerate reflecting the structure of the order parameter, and one of them is chosen as the symmetry breaking. For example, in the superradiant phase the spontaneous value of photons $\langle a \rangle$ and atom $\sum_{i=1}^N \langle S_i^+ \rangle$ appear, and the Floquet state in the set S corresponding to the symmetry broken state is realized. For the optical bistability which is shown in Sec. VI, two Floquet states exist in the set S in the thermodynamic limit. In Sec. V, we classify the stationary states according to this picture.

IV. MEAN-FIELD TREATMENT

In order to solve the time evolution, we need all the eigenvalues and eigenstates of the system [35]. It can be done only for systems with small degrees of freedom [27, 28, 36, 37]; It is difficult in the present case because the model of the cavity consists of many degrees of the freedom. As a solution of the difficulty, we focus on the nature of a cavity that all the atoms coherently interact with a single mode of photons, which causes an effective long-range interaction between atoms and study the time evolution of the system by a mean-field (MF) treatment. The MF treatment is rigorous in this case in the thermodynamic limit [38]. In the MF treatment, we assume that, for all the t , the density matrix is given by a product of density matrices of photons and atoms:

$$\rho_S(t) = \rho_p(t) \otimes \rho_a^{(1)}(t) \otimes \cdots \otimes \rho_a^{(N)}(t), \quad (34)$$

where $\rho_p(t)$ is the density matrix for photons and $\rho_a^{(i)}(t) (i = 1, \dots, N)$ is a 2×2 matrix for the density matrix of the i th atom. Here, we assume that all the atoms are in the same state $\rho_a(t)$. In this case the time evolutions of $\rho_p(t)$ and $\rho_a(t)$ are described by the Hartree-type equations. As a result, we can reduce the many-body problem to the one-body problem with MF parameters.

A. The master equation for photons

First we derive the master equation for photons. The equation is given by projecting out the degrees of freedom of atoms. We define the Hamiltonian for photons $\mathcal{H}_p(t)$ by replacing the operators of atoms into the average values.

$$\begin{aligned} \frac{\mathcal{H}_p(t)}{\hbar} &= \omega_p a^\dagger a + \frac{g}{\sqrt{N}} \sum_{i=1}^N [a (\langle S_i^+ \rangle + \chi \langle S_i^- \rangle) + a^\dagger (\langle S_i^- \rangle + \chi \langle S_i^+ \rangle)] \\ &\quad + 2\sqrt{N}\xi (a^\dagger + a) \cos(\omega_e t) + C(t). \end{aligned} \quad (35)$$

By introducing a dressed bosonic operator

$$b = a + \frac{1}{\omega_p} \left[\frac{g}{\sqrt{N}} \sum_{i=1}^N (\langle S_i^- \rangle + \chi \langle S_i^+ \rangle) + 2\sqrt{N}\xi \cos(\omega_e t) \right], \quad (36)$$

which satisfies the commutation relation $[b, b^\dagger] = 1$, this Hamiltonian can be rewritten as

$$\mathcal{H}_p(t) = \hbar\omega_p b^\dagger b + C'(t), \quad (37)$$

where $C(t)$ and $C'(t)$ are not the operators, and they do not contribute to the result. Thus, we redefine the Hamiltonian as

$$\mathcal{H}_p = \hbar\omega_p b^\dagger b. \quad (38)$$

This shows that the system is regarded to consist of the dressed photon field which incorporates the mean field of atoms. We derive the master equation of the cavity photon using Eq. (16),

$$\begin{aligned} \frac{\partial \rho_p(t)}{\partial t} &\simeq -i [\mathcal{H}_p(t), \rho_p(t)] \\ &\quad - \lambda^2 \int_0^\infty du \int_0^\infty d\omega \left| \tilde{k}_0(\omega) \right|^2 \\ &\quad \times \left\{ e^{-i\omega u} [X_0, \exp(-i\mathcal{H}_p(t)u) X_0 \exp(i\mathcal{H}_p(t)u) \rho_p(t)] \right. \\ &\quad \left. - e^{i\omega u} [X_0, \rho_p(t) \exp(-i\mathcal{H}_p(t)u) X_0 \exp(i\mathcal{H}_p(t)u)] \right\}, \\ &= -i [\mathcal{H}_p(t), \rho_p(t)] \\ &\quad - \lambda^2 \int_0^\infty d\omega \left| \tilde{k}_0(\omega) \right|^2 \\ &\quad \times \pi \{ \delta(\omega - \omega_p) ([a + a^\dagger, b\rho_p(t)] - [a + a^\dagger, \rho_p(t)b^\dagger]) \\ &\quad + \delta(\omega + \omega_p) ([a + a^\dagger, b^\dagger \rho_p(t)] - [a + a^\dagger, \rho_p(t)b]) \} \\ &\quad - i \left\{ P \left(\frac{1}{\omega - \omega_p} \right) [a + a^\dagger, b\rho_p(t) + \rho_p(t)b^\dagger] + P \left(\frac{1}{\omega + \omega_p} \right) [a + a^\dagger, b^\dagger \rho_p(t) + \rho_p(t)b] \right. \\ &\quad \left. + P \left(\frac{1}{\omega} \right) \frac{2}{\omega_p} \left[\frac{g}{\sqrt{N}} (1 + \chi) \sum_{i=1}^N (\langle S_i^- \rangle + \langle S_i^+ \rangle) + 4\sqrt{N}\xi \cos(\omega_e t) \right] [a + a^\dagger, \rho_p(t)] \right\}. \end{aligned} \quad (39)$$

Here, $|\tilde{k}_0(\omega)|^2 = \sum_{\omega_\alpha(>0)} |k_0(\omega_\alpha)|^2 \delta_{\omega_\alpha, \omega}$. Neglecting the contribution of the Cauchy principal value, we have

$$\frac{\partial \rho_p(t)}{\partial t} \simeq -i [\mathcal{H}_p(t), \rho_p(t)] - \kappa ([a + a^\dagger, b\rho_p(t)] - [a + a^\dagger, \rho_p(t)b^\dagger]), \quad (40)$$

where we introduce a parameter $\kappa = \pi\lambda^2 \left| \tilde{k}_0(\omega_p) \right|^2$.

B. The master equation for each atom

Next, we derive the master equation for each atom. Like the case of photon, the equation is given by projecting out the degrees of freedom of photons and other atoms. The Hamiltonian for each atom \mathcal{H}_a is given by

$$\mathcal{H}_a = \hbar\omega_a S_i^z + \frac{\hbar g}{\sqrt{N}} [(\langle a \rangle + \chi \langle a^\dagger \rangle) S_i^+ + (\langle a^\dagger \rangle + \chi \langle a \rangle) S_i^-]. \quad (41)$$

This Hamiltonian is diagonalized by a unitary operator U_a ,

$$\frac{1}{\hbar} U_a^\dagger \mathcal{H}_a U_a = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \quad (42)$$

where

$$\sigma \equiv \sqrt{\left(\frac{\omega_a}{2} \right)^2 + \frac{g^2}{N} |\langle a \rangle + \chi \langle a^\dagger \rangle|^2}, \quad (43)$$

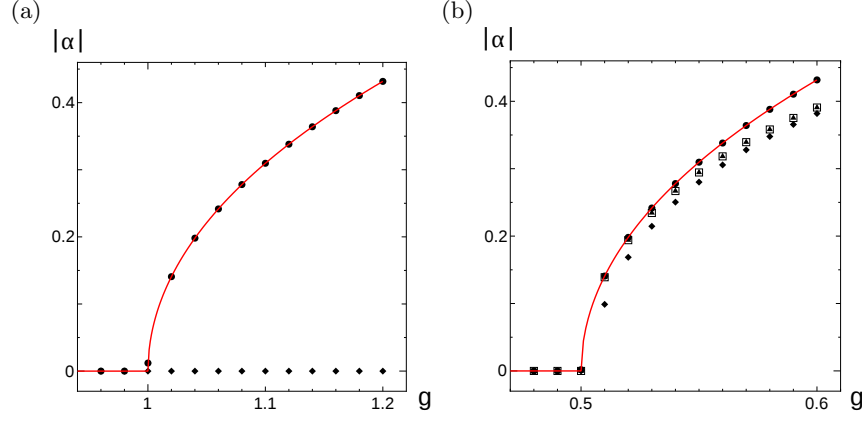


FIG. 1: (Color online) (a) Dependence of the ordered component $|\alpha| = |\langle a \rangle / \sqrt{N}|$ in the stationary states on the interaction strength g in the Tavis-Cummings model with $\omega_p = \omega_a = 1$. Bullets show results of the present treatment with $\kappa = \gamma = 0.1$ which agree with the exact results given by the red curve: $\alpha = \frac{1}{2} \sqrt{g^2 / \omega_p^2 - \omega_a^2 / g^2}$. Diamonds show the results of the treatment with simple Lindblad equation (7) with $\kappa = \gamma = 0.1$, where no ordered component appears at all. (b) Dependence of the ordered component $|\alpha|$ in stationary states on the interaction strength g in the Dicke model with $\omega_p = \omega_a = 1$. Bullets show results of the present treatment with $\kappa = \gamma = 0.1$ which agree with the exact results given by the red curve: $\alpha = \frac{1}{2} \sqrt{4g^2 / \omega_p^2 - \omega_a^2 / (4g^2)}$. Diamonds, squares, and triangles give the data obtained by the treatment with simple Lindblad equation (7) with $\kappa = \gamma = 0.1, 0.01$, and 0.001 , respectively. They converge to a limiting value as κ and γ decrease, but the value does not agree with the exact value.

and U_a is given by

$$U_a = \frac{1}{\sqrt{\eta^2 + \frac{g^2}{N} |\langle a \rangle + \chi \langle a^\dagger \rangle|^2}} \begin{pmatrix} \eta & -\frac{g}{\sqrt{N}} (\langle a \rangle + \chi \langle a^\dagger \rangle) \\ \frac{g}{\sqrt{N}} (\langle a^\dagger \rangle + \chi \langle a \rangle) & \eta \end{pmatrix}, \quad (44)$$

with

$$\eta \equiv \omega_a / 2 + \sqrt{(\omega_a / 2)^2 + g^2 |\langle a \rangle + \chi \langle a^\dagger \rangle|^2 / N}. \quad (45)$$

Here, we define $|+\rangle$ as eigenstate of σ and $|-\rangle$ as that of $-\sigma$. As was seen in the case of photon field, the system is regarded as consisting of a dressed atom, $\tilde{\mathbf{S}}_i (i = 1, \dots, N)$,

$$\begin{aligned} \mathcal{H}_a(t) &= 2\sigma U_a S_i^z U_a^\dagger \\ &= 2\sigma \tilde{S}_i^z, \end{aligned} \quad (46)$$

which incorporates the mean field of photon field. The master equation of the atom is given by

$$\begin{aligned} \frac{\partial \rho_a(t)}{\partial t} &= -i [\mathcal{H}_a(t), \rho_a(t)] \\ &\quad - \gamma \left\{ \langle - | U_a^\dagger (S_i^- + S_i^+) U_a | + \rangle [S_i^+ + S_i^-, \tilde{S}_i^- \rho_a(t)] \right. \\ &\quad \left. - \langle + | U_a^\dagger (S_i^- + S_i^+) U_a | - \rangle [S_i^+ + S_i^-, \rho_a(t) \tilde{S}_i^+] \right\}. \end{aligned} \quad (47)$$

Here, we introduced $\gamma \equiv \pi \lambda^2 |k_i (2\sigma)|^2$ as a constant because the energy scale of the bath is much larger than that of the interaction between the system and the bath. Like the case of the cavity photon, the relaxation term is related to the dressed atom $\tilde{\mathbf{S}}_i$.

C. New aspects of the present master equation

We derived the new types of master equations for photons Eq. (40) and each atom Eq. (47). Here we compare the present treatment and the treatment with the simple Lindblad equation (7) with $\gamma_z = 0$. In the treatment with the

simple Lindblad equation, the effect of dissipation for $\langle a \rangle$ is described by

$$\begin{aligned} \frac{d}{dt} \langle a \rangle_{\text{diss}} &= -\kappa \text{Tra} ([a^\dagger, a \rho_p] + [\rho_p a^\dagger, a]) \\ &= -\kappa \langle a \rangle. \end{aligned} \quad (48)$$

This means that the relaxation term causes simple reduction of the photon field at the constant rate κ . the effect of dissipation for $\langle S_i^z \rangle$ is given by

$$\begin{aligned} \frac{d}{dt} \langle S_i^z \rangle_{\text{diss}} &= -\gamma \text{Tr} S_i^z ([S_i^+, S_i^- \rho_S] + [\rho_S S_i^+, S_i^-]) \\ &= -2\gamma \left(\langle S_i^z \rangle + \frac{1}{2} \right). \end{aligned} \quad (49)$$

The number of the excited atoms relaxes to zero, which is expressed as $\langle S_i^z \rangle = -1/2$, at the constant rate 2γ . These damping terms are verified when the cavity photons and atoms are independent. But in the composite system of photons and atoms with large coupling constant g , they are not applicable because the system dissipates into the different state from $\langle a \rangle = 0$ and $\langle S_i^z \rangle = -1/2$.

In our treatment, the dissipation terms make the dressed photon field b and dressed atom \tilde{S}_i relax. It is the most important point that the relaxation is related to these dressed values, which relaxes the system into the genuine ground state. The effect of dissipation for the dressed photon field b is given by

$$\begin{aligned} \frac{d}{dt} \langle b \rangle_{\text{diss}} &= -\kappa \text{Tr} b ([a + a^\dagger, b \rho_S(t)] - [a + a^\dagger, \rho_S(t) b^\dagger]) \\ &= -\kappa (\langle b \rangle - \langle b^\dagger \rangle). \end{aligned} \quad (50)$$

This shows the relaxation of the dressed photon field. The effect of dissipation for the dressed atom \tilde{S}_i^z is given by

$$\begin{aligned} \frac{d}{dt} \langle \tilde{S}_i^z \rangle_{\text{diss}} &= -\gamma \text{Tr} \tilde{S}_i^z \left(\langle -| U_a^\dagger (S_i^- + S_i^+) U_a | + \rangle [S_i^+ + S_i^-, \tilde{S}_i^- \rho_S(t)] + \text{h.c.} \right) \\ &= -2\gamma_{[g, \langle a \rangle]} \left(\langle \tilde{S}_i^z \rangle + \frac{1}{2} \right), \end{aligned} \quad (51)$$

which describes the relaxation of an excited dressed atom. Here, the coupling constant γ is also modulated by the mean field as

$$\gamma_{[g, \langle a \rangle]} \equiv | \langle -| U_a^\dagger (S_i^- + S_i^+) U_a | + \rangle |^2 \gamma. \quad (52)$$

With these dissipation terms, the system indeed relaxes to the true ground state of the Hamiltonian in the case of no external field ($\xi = 0$). We show the dependences of the ordered component $|\alpha| = |\langle a \rangle| / \sqrt{N}$ in the stationary states on the interaction strength g of the Tavis-Cummings model and the Dicke model in Fig. 1(a) and (b), respectively. As we see, the results of the present treatment (bullets) reproduce the equilibrium states (red curves). In the figures, we also give the results obtained by the treatment with the Lindblad equation (7) with various strength of the couplings, $\kappa = \gamma = 0.1$ (diamonds), 0.01 (squares), and 0.001 (triangles). In the case of the Tavis-Cummings model (Fig. 1(a)), the equation (7) does not show any ordered state. In the case of the Dicke model (Fig. 1(b)), the simplified equation (7) gives a phase transition, but fails to give the correct values of the order parameter.

Thus, the present formalism of dissipation terms satisfies the minimal condition for the study of the strong interaction cases. The terms are regarded to be the extension of the master equation with dressed modes. Therefore we use this formalism in the present study.

D. The equations of motions

We express the master equation with the scaled quantities,

$$\alpha = \frac{\langle a \rangle}{\sqrt{N}}, \quad m^z = \frac{1}{N} \sum_{i=1}^N \langle S_i^z \rangle, \quad m^\pm = \frac{1}{N} \sum_{i=1}^N \langle S_i^\pm \rangle. \quad (53)$$

The equations of motions for α , m^+ , and m^z are given by multiplying operator a , S_i^+ , and S_i^z respectively on the corresponding master equations Eq. (40) and Eq. (47), respectively. As a result, we obtain a set of classical equations of motions: for the photon field,

$$\begin{aligned} \frac{\partial \alpha}{\partial t} = & -i(\omega_p \alpha + g(m^- + \chi m^+) + 2\xi \cos(\omega_e t)) \\ & - \kappa \left[\left\{ \alpha + \frac{g}{\omega_p} (m^- + \chi m^+) \right\} - \left\{ \alpha^* + \frac{g}{\omega_p} (m^+ + \chi m^-) \right\} \right], \end{aligned} \quad (54)$$

for the dipole moment of atoms,

$$\begin{aligned} \frac{\partial m^+}{\partial t} = & -2i \left\{ -\frac{\omega_a}{2} m^+ + g(\alpha^* + \chi \alpha) m^z \right\} \\ & - \frac{\gamma}{\left\{ \eta^2 + g^2 |\alpha + \chi \alpha^*|^2 \right\}^2} \left[-\left\{ \eta^2 - g^2 (\alpha^* + \chi \alpha)^2 \right\} \left\{ \eta g (\alpha + \chi \alpha^*) + g^2 (\alpha + \chi \alpha^*)^2 m^+ + \eta^2 m^- \right\} \right. \\ & \left. + \left\{ \eta^2 - g^2 (\alpha + \chi \alpha^*)^2 \right\} \left\{ \eta g (\alpha^* + \chi \alpha) + \eta^2 m^+ + g^2 (\alpha^* + \chi \alpha)^2 m^- \right\} \right], \end{aligned} \quad (55)$$

and for the number of the excited atoms,

$$\begin{aligned} \frac{\partial m^z}{\partial t} = & -ig \left\{ (\alpha + \chi \alpha^*) m^+ - (\alpha^* + \chi \alpha) m^- \right\} \\ & - \frac{\gamma}{\left\{ \eta^2 + g^2 |\alpha + \chi \alpha^*|^2 \right\}^2} \\ & \times \left[\left\{ \eta^2 - g^2 (\alpha^* + \chi \alpha)^2 \right\} \left\{ \eta g (\alpha + \chi \alpha^*) (m^+ + m^-) - g^2 (\alpha + \chi \alpha^*)^2 \left(m^z - \frac{1}{2} \right) + \eta^2 \left(m^z + \frac{1}{2} \right) \right\} \right. \\ & \left. + \left\{ \eta^2 - g^2 (\alpha + \chi \alpha^*)^2 \right\} \left\{ \eta g (\alpha^* + \chi \alpha) (m^+ + m^-) + \eta^2 \left(m^z + \frac{1}{2} \right) - g^2 (\alpha^* + \chi \alpha)^2 \left(m^z - \frac{1}{2} \right) \right\} \right]. \end{aligned} \quad (56)$$

In the region, $g \ll \omega_a$, $g \ll \omega_p$, and $\xi \ll \omega_p$, with the rotating wave approximation for the interaction term between the system and the bath, these equations are reduced into those obtained by the simple Lindblad equation (7). These equations seem to be complex, but the most important thing here is that the effect of the atom-photon coupling is incorporated into the relaxation terms. The nonlinearity of the equations increases with $g\alpha$, which causes the Dicke transition and complicated phenomena in large ξ . In the next section, we discuss the behavior of the stationary states.

V. TIME-EVOLUTION AND TYPES OF STATIONARY STATES

The stationary state in thermodynamic limit can be obtained by evolving the state by the equations of motion (54), (55), and (56) for a long time instead of obtaining the floquet state $\rho_{S,SS}(t)$ explicitly. We concentrate on the case $\omega_a = \omega_p = \omega_e \equiv \omega$ which is set to be one and $\kappa = \gamma$ which is set to be 0.1. We solve the equations by means of Runge-Kutta method. In the present study, we set time interval $\Delta t = 0.0002\pi$ which is fine enough for the present simulation. We confirm that this value of Δt is enough small by changing Δt . We performed simulations with increments $\Delta g = 0.02$ and $\Delta \xi = 0.001$. To observe the stationary state, we study the behavior in the large t region, e.g., $(t = 9800\pi - 10000\pi)$. We confirmed that the state reached a stationary state by changing the range of t .

In order to classify the stationary states, we make use of the Fourier transformation to detect this symmetry,

$$\alpha_k = \frac{1}{200\pi} \int_{9800\pi}^{10000\pi} dt' \alpha(t') e^{ik\omega_e t'} \quad (k = 0.01m \mid m \in \mathbf{Z}, -210 \leq m \leq 210). \quad (57)$$

According to Eq. (30), if the stationary state is unique,

$$\overline{f(\alpha, \alpha^*, m^\pm, m^z)} = \overline{f(-\alpha, -\alpha^*, -m^\pm, m^z)}, \quad (58)$$

and only the odd numbers of k in Eq. (57) are allowed:

$$\alpha(t) = \sum_{k=(\text{odd numbers})} \alpha_k \exp(ik\omega_e t). \quad (59)$$

On the other hand, if the symmetry is broken, then the even numbers are allowed, too:

$$\alpha(t) = \sum_{k=(\text{integers})} \alpha_k \exp(ik\omega_e t). \quad (60)$$

We call the static component α_0 “spontaneous order of photon”.

When the symmetry is not broken, α_k has delta peaks at $k = \pm 1$. This phase represents that the system is only driven by external field and we call this “regularly oscillating phase”. If the symmetry is broken, we observe the delta peaks at $k = 0$ and $k = \pm 2$ in addition to those at $k = \pm 1$. In this phase, the spontaneous order of photon α_0 appears and we call this phase “ordered phase”. Besides these phases, there are regions which do not belong to the above two phases. In these regions, the Fourier spectrum has delta peaks at non-integer k , or even shows broad peaks. We call this “irregularly oscillating phase”. For each Fourier component k , we set the threshold for the existence of delta peak to be $|\alpha_k|^2 = 1.0 \times 10^{-4}$.

In experiments, we can measure the photon emission from the cavity which represents the energy transfer per unit time from the cavity to the thermal photon bath. It depends on the dressed value (see Appendix B) given by

$$I_p = 4\kappa\omega_p(\text{Im}b)^2. \quad (61)$$

It is possible to classify three phases by the photon emission. In the regularly oscillating phase, the Fourier coefficients have delta peaks at $k = 0$ and $k = \pm 2$. In the ordered phase, they have delta peaks at $k = \pm 1$ in addition. In the irregularly oscillating phase, they also have delta peaks at non-integer k , or even shows broad peaks.

VI. PHASE DIAGRAM OF THE STATIONARY STATES OF TAVIS-CUMMINGS MODEL

In this section, we study the phase diagram in the Tavis-Cummings model driven by the AC external field in the coordinate of the strength of interaction g and the intensity of external field ξ . Here, we show two results: First, we confirm the optical bistability to appear in the region $g \sim \omega$ and $\xi \sim \omega$, which has been studied in the region where $g \ll \omega$ and $\xi \ll \omega$ [20, 21, 26]. Second, we observe the irregular oscillating phase in the region where g and ξ are strong.

We study the stationary state when ξ is increased and when ξ is decreased at a fixed value of g , respectively. When we change ξ , as the initial state we use the stationary state of α , m^z , and m^+ for the previous value of ξ . We divide the phase diagram into three phases: regularly oscillating phase, ordered phase, and irregularly oscillating phase.

From now, we will explain the phase diagram in Fig. 2 in the following three subsections. In the first section VIA, we will study the region where the optical bistability appears. In this region, the system is in the regularly oscillating phase. In the second section VIB, we will explain the strongly interacting and weakly excited region. In this region, the effect of interaction between atoms through photon field is dominant. Therefore, we find an ordered phase due to the strong interaction g (Dicke transition). Finally, in the third section VIC, we will show the behavior in the highly interacting and highly excited region. Here, we find the irregularly oscillating phase.

A. Optical Bistability

In the region where g and ξ are weak ($g\xi < 0.07$), we find only the regularly oscillating phase. In this phase, the photon amplitude α oscillates around the origin and the dipole moment of atoms oscillates around the origin in the (m^x, m^y) plane.

In Fig. 3 (a), we show the ξ -dependence of the photon emission I_p averaged over the period of the AC external field in a case of very weak interaction $g = 0.12$. In this figure, $\overline{I_p}$ smoothly changes with increasing and decreasing ξ .

In Fig. 3 (b), we also show the ξ -dependence of $\overline{I_p}$ for $g = 0.4$, where we confirm the optical bistability to appear; We find a discontinuous transition in each of the ξ -increasing process and in the ξ -decreasing process, and the points where the transitions appear are different. Namely, we find a hysteresis phenomenon. As for $\overline{m^z}$, $\overline{m^z}$ jumps up from -0.2 to around zero in the ξ -increasing process and jumps back in the ξ -decreasing process at the different points from that in the ξ -increasing process. It is the transition from a regularly oscillating phase to another regularly oscillating phase; the static order parameters do not appear. The optical bistability is a kind of non-equilibrium phase transitions

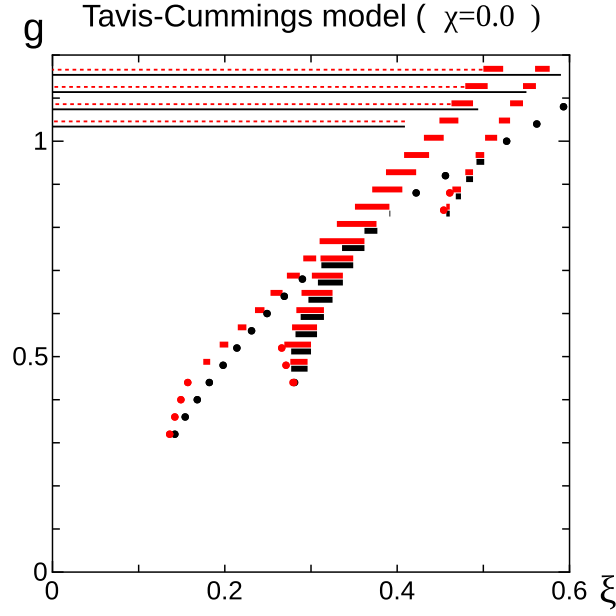


FIG. 2: (Color online) The phase diagram of the Tavis-Cummings model is parameterized by the interaction strength g and the intensity of the external field ξ . This phase diagram shows the steady states in the processes of increasing or decreasing ξ at the fixed value of g for the case of $\omega_a = \omega_p = \omega_e = 1$. (i) The black bullets and red bullets denote points of discontinuous changes of the state in the ξ -increasing and ξ -decreasing processes, respectively (see Sec. VIA). The optical bistability takes place between them. (ii) The regularly oscillating phase is denoted by blank space (see Sec. VIA). (iii) The ordered phases in the ξ -increasing and ξ -decreasing processes are denoted by black thin lines and a red dotted lines, respectively (see Sec. VIB). (iv) The irregularly oscillating phases in the ξ -increasing and ξ -decreasing processes is denoted by black bold lines and red bold lines, respectively (see Sec. VIC).

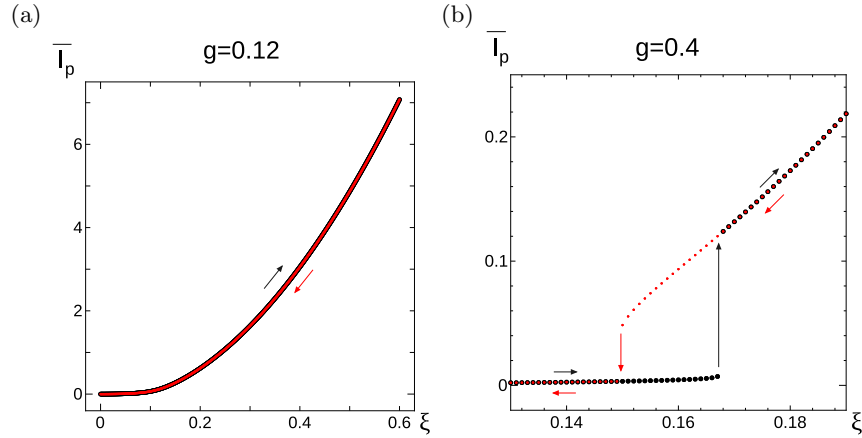


FIG. 3: (Color online) Dependence of the energy dissipation \overline{T}_p on ξ for $g = 0.12$ (a) and for $g = 0.4$.

originated in the relaxation process. The external field continuously pumps photons which are to be relaxed through two processes, one of them is via a direct photon relaxation processes, and the other is via the atoms. As we find in Eqs. (54) and (55), when g is small compared to ω_a and ω_p , the relaxation of photon via the direct photon relaxation is proportional to $\kappa\alpha$, while the relaxation of photon via atoms is of order $g^2\alpha m^z/\gamma$. Therefore, if the external field is weak, both the relaxation processes contribute because $m^z \approx -1/2$, because the system is still near the ground state. When the external field is strongly applied, it is mainly through the photon emission because the energy emission via atoms is suppressed and m^z approaches to zero. Between these cases the mechanism changes discontinuously.

B. Ground state phase transition due to the interaction g

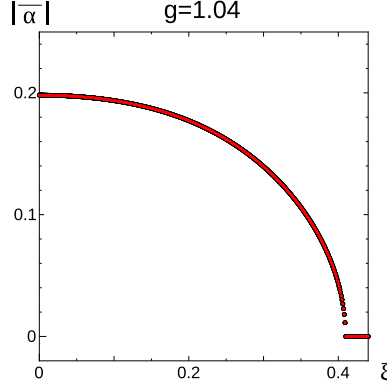


FIG. 4: (Color online) Dependence of the ordered component $|\bar{\alpha}|$ on ξ for $g = 1.04$.

When g is larger than the critical value $g_C^{\text{TC}} = \sqrt{\omega_a \omega_p} = 1$, the photon field and dipole moments of atoms appear spontaneously as the Dicke transition [17, 18]. In Fig. 4, we show the ξ -dependence of the order parameter $|\bar{\alpha}|$ for $g = 1.04$. We find that the ordered component of photon disappears continuously. When ξ is increased, the ordered component of atoms $|\overline{m^+}|$ also disappears continuously, and m^z approaches to -0.5 . In both processes of increasing and decreasing ξ , we do not find hysteresis. In this case, the AC external field disturbs the order, which is plausible because the external field excites the system from the ground state.

In the phase diagram, we draw the black thin lines in the ξ -increasing processes and the red dotted lines in the ξ -decreasing processes, where the ordered component appears.

C. Highly excited and highly interacting region

In the region where g and ξ are strong ($g\xi > 0.07$), we find the irregularly oscillating phase. In this phase, we do not observe the order parameter. In Fig. 5, we show the time evolution of I_p for $(\xi, g) = (0.37, 0.86)$.

In the phase diagram (Fig. 2), we draw black bold lines in the ξ -increasing process and red bold lines in the ξ -decreasing process, where the irregularly oscillating phase appears. On the phase boundary between the irregularly oscillating phase and the regularly oscillating phase, g is almost proportional to ξ . The detailed analysis of this property will be studied in the future work.

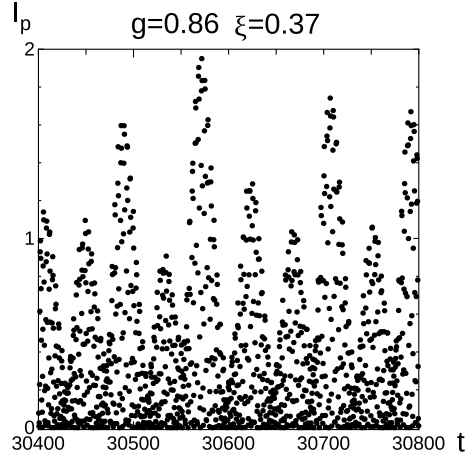


FIG. 5: (Color online) Time evolution of I_p in the region of irregularly oscillating at $\xi = 0.37$ for $g = 0.86$.

VII. PHASE DIAGRAM OF THE STATIONARY STATES OF THE DICKE MODEL

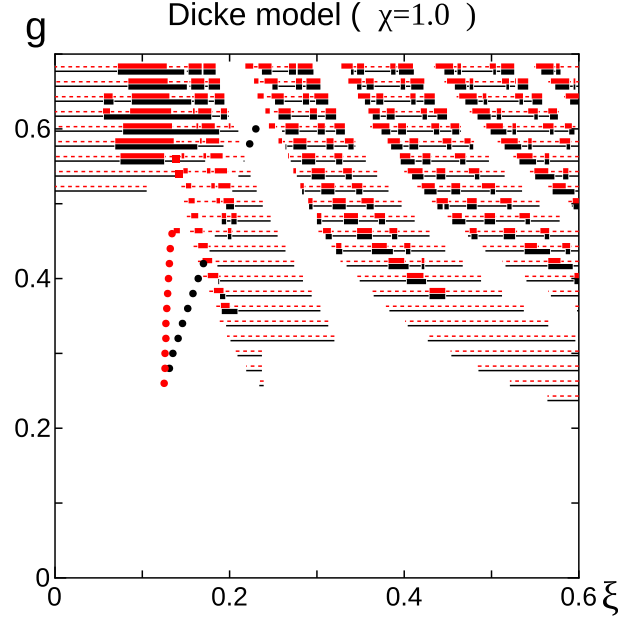


FIG. 6: (Color online) The phase diagram of the Dicke model is parameterized by the interaction strength g and the intensity of the external field ξ . This phase diagram shows the phase of the stationary state in the process of increasing and decreasing ξ at each fixed value of g for the case of $\omega_a = \omega_p = \omega_e = 1$. (i) The black bullets and red bullets denote points where the stationary state changes discontinuously from the regularly oscillating phase to another regularly oscillating phase in the increasing and decreasing ξ processes, respectively. The red squares observed around $(\xi, g) = (0.13, 0.54)$ show the discontinuous jump from the ordered phase to another ordered phase in the ξ -decreasing process. (ii) The regularly oscillating phase is denoted by the blank space. (iii) The ordered phase in the ξ -increasing and ξ -decreasing processes are denoted by black thin lines and red dotted lines, respectively (see Sec. VII A). (iv) The irregularly oscillating phase in the increasing and decreasing ξ processes are denoted by black bold lines and red bold lines, respectively (see Sec. VII A).

Next, we will study the phase diagram in the Dicke model driven by the AC external field characterized by the strength of interaction g and the intensity of external field ξ . In this section, we show the two following results: First, the rotating wave approximation is justified when g and ξ are weak, while it is not justified when g and ξ are strong. Second, instead of the irregularly oscillating phase in the Tavis-Cummings model, we find a novel symmetry breaking phenomenon due to the AC external field when g and ξ are strong.

We also divide the phase diagram into three phases: regularly oscillating phase, ordered phase, and irregularly oscillating phase as we saw in the case of the Dicke model. In Fig. 6, we give the phase diagram in the same notation of Fig. 2.

From now we will explain the phase diagram. When the atom-photon coupling and the intensity of the AC external field are small ($g\xi < 0.06$), we observe a similar feature in the phase diagram to the case of the Tavis-Cummings model; The region belongs to the regularly oscillating phase. When the interaction is stronger, we find the optical bistability. It shows that the rotating wave approximation is justified and that the effect of irrotation term qualitatively does not appear in this region.

We also find the similar behavior in the region where the interaction is strong and the external field is weak. When the interaction exceeds $g_D = \sqrt{\omega_a \omega_p}/2$, the spontaneous order parameters $|\bar{\alpha}|$ and $|\bar{m}^+|$ appear [17, 18]. When ξ is increased, the order parameters decrease gradually and disappear continuously to zero as we saw in the case of the Tavis-Cummings model.

A. Ordered state of driven component in the highly excited and highly interacting region

In contrast to the Tavis-Cummings model, the ordered state of driven component appears in the highly excited and highly interacting region. In Fig. 7 (a) and (b), we show the ξ -dependences of the order parameters $|\bar{\alpha}|$ and

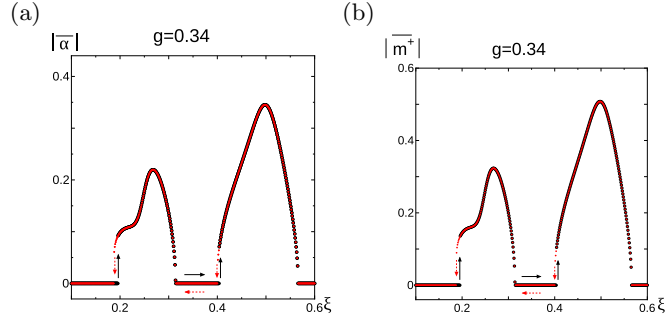


FIG. 7: (Color online) Dependence of quantities on ξ in the region $0.1 < \xi < 0.6$ for $g = 0.34$. (a) The energy dissipation $|\bar{\alpha}|$ on ξ , (b) the dipole moment $|\bar{m}^+|$ on ξ .

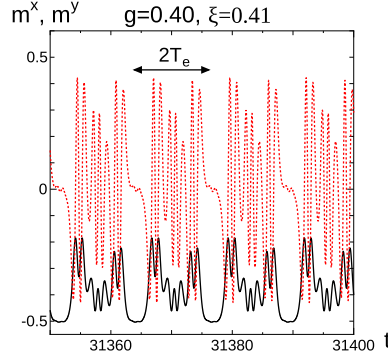


FIG. 8: (Color online) Time dependences of m^x (black bold line) and m^y (red dotted line). The period of the system is twice as long as that of the AC external field.

$|\bar{m}^+|$. Order parameters appear and disappear repeatedly and $|\bar{\alpha}|$ and $|\bar{m}^+|$ behave in the same way. In the ξ -increasing process, order parameters appear discontinuously and disappear continuously. In the ξ -decreasing process, they exhibit the same way. In the both processes, the continuous transitions appear at the same value of ξ and the discontinuous transitions appear at the different value of ξ . From the argument given in Sec. V, the finite order parameters indicate the spontaneous symmetry breaking. In the region photon field α and the dipole moment of atoms m^+ oscillate around the finite value. Here, it should be noted that the value of $|\bar{m}^+|$ exceeds the bound around $\xi = 0.5$, which should be attributed to approximations in the derivation of the master equation. This problem will be studied elsewhere.

Next, we show the behavior of the irregularly oscillating phase. As we saw in Sec. V, “irregular” means that the system does not have the same period of the external field. In Fig. 8, we show the time dependences on m^x (black bold line) and m^y (red dotted line) for $g = 0.4$ and $\xi = 0.41$. We find that the period is twice as long as that of the AC external field, and m^x is always negative, which shows the symmetry breaking.

In the phase diagram, the repeated belt-like structure appears which indicates that the AC external field makes the order parameter align at the discrete values of ξ . On the phase boundary, g is almost inversely proportional to ξ , which will be studied in the future work.

VIII. SUMMARY AND DISCUSSION

In the present paper, we studied the cooperative phenomena of photons and atoms under the AC external field. In order to study the stationary state, we used the formalism of a master equation for open systems. The Lindblad equation (7) is not appropriate when the atom-photon coupling is strong compared to the energy of photons and atoms. We derived the master equation which incorporates the atom-photon coupling into the dissipation terms with the use of the mean-field treatment under an approximation that the correlation time of the thermal bath is regarded

to be short. The dissipation terms of the present master equation are related to dressed quantities; The dissipation terms relax the dressed photon b which incorporates the mean field produced by atoms and relax the dressed atom \tilde{S}_i^z which incorporates the mean field produced by photons. We studied the stationary state without the external field ($\xi = 0$) and confirmed that the dissipation terms reproduce the Dicke transition correctly.

We showed the following results. We presented the phase diagrams parameterized by the strength of the atom-photon coupling g and the intensity of the AC external field ξ for the Tavis-Cummings model and the Dicke model under the AC external field. For small values of the product $g\xi$, we found the similar behavior in the both phase diagrams. For small g , the system belongs to the regularly oscillating phase and we observed no singular behavior. When g increases a little, the optical bistability appears while changing the value of ξ . When g exceeds the critical value, the spontaneous order parameters appear as the Dicke transition. In the region where the external field is weak, the AC external field disturbs the order. For large values of the product $g\xi$, we found the different behavior in the phase diagrams for the Tavis-Cummings model and the Dicke model and therefore the rotating wave approximation is not appropriate in this region. In the case of the Tavis-Cummings model, the irregularly oscillating phase appeared. On the phase boundary between the irregularly oscillating phase and the regularly oscillating phase, g was almost proportional to ξ . In the case of the Dicke model, the repeated belt-like structure of the ordered phase appeared instead of the irregularly oscillating phase, which describes a non-equilibrium phase transition with a spontaneous symmetry breaking. On the phase boundary between the ordered phase and the regularly oscillating phase, g was almost inversely proportional to ξ .

The detailed analysis on the phase boundary which is approximately given by a line $g/\xi = \text{constant}$ in the Tavis-Cummings model and by a line given by a line $g\xi = \text{constant}$ in the Dicke model and the mechanism how the ordered states appear under the AC external field will be reported elsewhere.

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Appendix A: The simple Lindblad equation

In this section, we derive the simple Lindblad equation (7) from Eq. (16).

$$\begin{aligned} \frac{\partial \rho_S(t)}{\partial t} = & -\frac{i}{\hbar} [\mathcal{H}_S(t), \rho_S(t)] \\ & - \lambda^2 \sum_{n=0}^N \int_0^\infty du \int_0^\infty d\omega \left| \tilde{k}_n(\omega) \right|^2 \\ & \times (e^{-i\omega u} [X_n, e^{-i\mathcal{H}_S(t)u} X_n e^{i\mathcal{H}_S(t)u} \rho_S(t)] - e^{i\omega u} [X_n, \rho_S(t) e^{-i\mathcal{H}_S(t)u} X_n e^{i\mathcal{H}_S(t)u}]). \end{aligned} \quad (\text{A1})$$

First, we drop the irrotational terms in the interaction Hamiltonian between the system and the thermal bath. It is expected not to be essential because the interaction strength between the bath and the system is weak. We then obtain

$$\begin{aligned} \frac{\partial \rho_S(t)}{\partial t} = & -\frac{i}{\hbar} [\mathcal{H}_S(t), \rho_S(t)] \\ & - \lambda^2 \sum_{n=0}^N \int_0^\infty du \int_0^\infty d\omega \left| \tilde{k}_n(\omega) \right|^2 \\ & \times \left\{ e^{-i\omega u} \left([X_n^+, e^{-i\mathcal{H}_S(t)u} X_n^- e^{i\mathcal{H}_S(t)u} \rho_S(t)] + [X_n^-, e^{-i\mathcal{H}_S(t)u} X_n^+ e^{i\mathcal{H}_S(t)u} \rho_S(t)] \right) \right. \\ & \left. - e^{i\omega u} \left([X_n^+, \rho_S(t) e^{-i\mathcal{H}_S(t)u} X_n^- e^{i\mathcal{H}_S(t)u}] + [X_n^-, e^{-i\mathcal{H}_S(t)u} X_n^+ e^{i\mathcal{H}_S(t)u} \rho_S(t)] \right) \right\}, \end{aligned} \quad (\text{A2})$$

where

$$X_0^- = a, X_0^+ = a^\dagger, \quad (\text{A3})$$

$$X_i^- = S_i^-, X_i^+ = S_i^+ \quad (i = 1 \dots N). \quad (\text{A4})$$

Next, we drop the atom-photon coupling and the external field in the system Hamiltonian in the relaxation terms:

$$\mathcal{H}_S(t) \rightarrow \omega_p a^\dagger a + \sum_{i=1}^N \omega_a S_i^z, \quad (\text{A5})$$

which is justified in the region $g \ll \omega_a$, $g \ll \omega_p$, and $\xi \ll \omega_p$. If we neglect the contribution of the Cauchy principal value, we obtain the simple Lindblad equation (7) given by

$$\frac{\partial \rho_S(t)}{\partial t} = -\frac{i}{\hbar} [\mathcal{H}_S(t), \rho_S(t)] - \kappa ([a^\dagger, a \rho_S(t)] + [\rho_S(t) a^\dagger, a]) - \gamma \sum_{i=1}^N ([S_i^+, S_i^- \rho_S(t)] + [\rho_S(t) S_i^+, S_i^-]), \quad (\text{A6})$$

where $\kappa = \pi \lambda^2 |\tilde{k}_0(\omega_p)|^2$ and $\gamma = \pi \lambda^2 |\tilde{k}_i(\omega_a)|^2$,

It is noted that we can not neglect the effect of the interaction term and the external field easily because $\omega_a - \omega_e$ and $\omega_p - \omega_e$ are important in the stationary driven state. In the resonant case where $\omega_e = \omega_a = \omega_p$, the system indeed shows the optical bistability as a cooperative phenomena via atom-photon coupling.

Appendix B: the evaluation of the intensity of the transmission light

In this paper we classify the stationary states into three phases. It is possible to distinguish them by the fourier spectrum of the intensity of the transmission light from the cavity I_p . In this Appendix, we derive the specific form of I_p . We consider the time development of the energy of the system $E(t)$, which is given by the master equation:

$$\begin{aligned} \frac{\partial}{\partial t} E(t) &= \text{Tr} \frac{\partial \mathcal{H}_S(t)}{\partial t} \rho_S(t) \\ &- \sum_{n=0}^N \int_0^\infty du \int_0^\infty d\omega \lambda^2 |\tilde{k}_n(\omega)|^2 \text{Tr} \mathcal{H}_S(t) (e^{-i\omega u} [X_n, X_n(t, t-u) \rho_S(t)] - e^{i\omega u} [X_n, \rho_S(t) X_n(t, t-u)]) \end{aligned} \quad (\text{B1})$$

The first term represents the energy transfer by the external field and the second term represents the energy dissipation. The intensity of the transmission light I_p corresponds to the $n = 0$ term of the energy dissipation,

$$N I_p = \int_0^\infty du \int_0^\infty d\omega \lambda^2 |\tilde{k}_0(\omega)|^2 \text{Tr} \mathcal{H}_S(t) (e^{-i\omega u} [X_0, X_0(t, t-u) \rho_S(t)] - e^{i\omega u} [X_0, \rho_S(t) X_0(t, t-u)]), \quad (\text{B2})$$

which represents the energy dissipation from the system to the photon bath. After making the mean-field treatment and regarding the external field as a constant during the time evolution from $t - u$ to t ,

$$\begin{aligned} N I_p &= \int_0^\infty du \int_0^\infty d\omega \lambda^2 |\tilde{k}_0(\omega)|^2 \text{Tr}_{\text{photon}} (e^{-i\omega u} \mathcal{H}_p(t) [a + a^\dagger, e^{-i\mathcal{H}_p(t)u} (a + a^\dagger) e^{i\mathcal{H}_p(t)u} \rho_p(t)] \\ &\quad - e^{i\omega u} \mathcal{H}_p(t) [a + a^\dagger, \rho_p(t) e^{-i\mathcal{H}_p(t)u} (a + a^\dagger) e^{i\mathcal{H}_p(t)u}]) \\ &= \kappa \text{Tr}_{\text{photon}} \mathcal{H}_p(t) ([a + a^\dagger, b \rho_p(t)] - [a + a^\dagger, \rho_p(t) b^\dagger]), \\ &= 4\kappa \omega_p (\text{Im} \langle b \rangle)^2, \end{aligned} \quad (\text{B3})$$

where $\kappa = \pi \lambda^2 |\tilde{k}_0(\omega_p)|^2$ and $b = a + [g \sum_{i=1}^N (\langle S_i^- \rangle + \chi \langle S_i^+ \rangle) / \sqrt{N} + 2\sqrt{N} \xi \cos(\omega_e t)] / \omega_p$. In the last transformation, we approximate that photon state is coherent. With the scaled quantities, I_p is rewritten as

$$I_p = 4\kappa \omega_p \left\{ \text{Im} \left[\alpha + \frac{g}{\omega_p} (m^- + \chi m^+) \right] \right\}^2. \quad (\text{B4})$$

Appendix C: The effect of the thermal bath coupling with all the atoms uniformly

In this Appendix, we study the dependences of stationary states on the type of dissipative environment which consists of a global-coupling bath and local-coupling baths. The global-coupling bath is given by the following

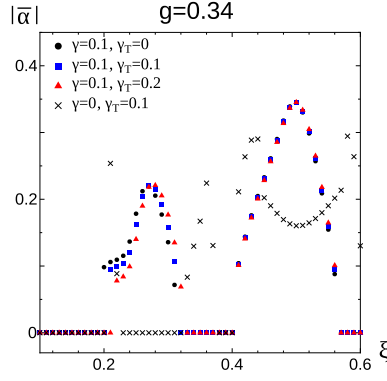


FIG. 9: (Color online) Dependences of the order parameter of photons $|\bar{\alpha}|$ on ξ in the ξ -increasing process for the case of the Dicke model ($\chi = 1$). We use black circles for $\gamma = 0.1$ and $\gamma_T = 0$ (Fig. 7), blue squares for $\gamma = 0.1$ and $\gamma_T = 0.1$, and red triangles for $\gamma = 0.1$ and $\gamma_T = 0.2$. We find the qualitatively similar behavior in these different types of dissipative environments. We also plot the points for $\gamma = 0$ and $\gamma_T = 0.1$ by crosses and we find the different behavior from other cases.

Hamiltonian

$$\mathcal{H}^T = \sum_{\omega_\alpha(>0)} \omega_\alpha A_T^\dagger(\omega_\alpha) A_T(\omega_\alpha) + \lambda \sum_{i=1}^N (S_i^+ + S_i^-) \sum_{\omega_\alpha(>0)} (k_T^*(\omega_\alpha) A(\omega_\alpha) + k_T(\omega_\alpha) A^\dagger(\omega_\alpha)), \quad (\text{C1})$$

where the first term represents the free boson bath and the second term represents the interaction between all atoms in the system and the thermal bath. Here, $A_T^\dagger(\omega_\alpha)$ and $A_T(\omega_\alpha)$ are creation and annihilation operators of the thermal bath, respectively. In the same manner as in Sec IV B, we obtain the contribution of the global-coupling bath to the equation motions of atoms by tracing out the degrees of the freedom of the thermal bath: for the dipole moment of atoms,

$$\begin{aligned} \frac{\partial}{\partial t} m_{\text{diss}}^+ &= - \frac{2\gamma_T}{(\eta^2 + g^2|\alpha + \chi\alpha^*|^2)^2} m^z \\ &\times [(\eta^2 - g^2(\alpha^* + \chi\alpha)^2)(\eta^2 m^- - g^2(\alpha + \chi\alpha^*)^2 m^+ - 2\eta g(\alpha + \chi\alpha^*) m^z) \\ &- (\eta^2 - g^2(\alpha + \chi\alpha^*)^2)(\eta^2 m^+ - g^2(\alpha + \chi\alpha^*)^2 m^- - 2\eta g(\alpha^* + \chi\alpha) m^z)] \end{aligned} \quad (\text{C2})$$

and for the number of the excited atoms,

$$\begin{aligned} \frac{\partial}{\partial t} m_{\text{diss}}^z &= - \frac{\gamma_T}{(\eta^2 + g^2|\alpha + \chi\alpha^*|^2)^2} (m^+ - m^-) \\ &\times [(\eta^2 - g^2(\alpha^* + \chi\alpha)^2)(\eta^2 m^- - g^2(\alpha + \chi\alpha^*)^2 m^+ - 2\eta g(\alpha + \chi\alpha^*) m^z) \\ &- (\eta^2 - g^2(\alpha + \chi\alpha^*)^2)(\eta^2 m^+ - g^2(\alpha + \chi\alpha^*)^2 m^- - 2\eta g(\alpha^* + \chi\alpha) m^z)], \end{aligned} \quad (\text{C3})$$

where $\gamma_T = \sum_{\omega_\alpha(>0)} |k_T(\omega_\alpha)|^2 \delta_{\omega_\alpha, 2\sigma}$.

In Fig. 9, we show the stationary states with various kinds of the coupling strength between the system and the thermal baths. Here, γ and γ_T denote the coupling strengths of the local-coupling bath and that of the global-coupling bath, respectively. We show the stationary states for $\gamma = 0.1$ and $\gamma_T = 0$ by black circles, $\gamma = 0.1$ and $\gamma_T = 0.1$ by blue squares, $\gamma = 0.1$ and $\gamma_T = 0.2$ by red triangles, and $\gamma = 0$ and $\gamma_T = 0.1$ by crosses. When a dissipative environment only consists of the global-coupling bath (crosses), the total spin is conserved, and thus we find a qualitative different behavior from the cases with non-zero γ . On the other hand, when the total spin is not conserved ($\gamma \neq 0$), we find a similar dependence. We also checked the effect of the global-coupling bath (γ_T) does not cause qualitative change in other parameter regions.

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- [1] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51**, 89 (1963).
 - [2] B. W. Shore and P. L. Knight, J. Mod. Opt. **40**, 1195 (1993).

- [3] M. Tavis and F. W. Cummings, Phys. Rev. **170**, 379 (1968).
- [4] G. S. Agarwal, Phys. Rev. Lett. **53**, 1732 (1984).
- [5] Y. Kaluzny, P. Goy, M. Gross, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **51**, 1175 (1983).
- [6] M. G. Raizen, R. J. Thompson, R. J. Brecha, H. J. Kimble, and H. J. Carmichael, Phys. Rev. Lett. **63**, 240 (1989).
- [7] Y. Zhu, D. J. Gauthier, S. E. Morin, Q. Wu, H. J. Carmichael, and T. W. Mossberg, Phys. Rev. Lett. **64**, 2499 (1990).
- [8] A. Wallraff, D. Schuster, A. Blais, L. Frunzio, R. Huang, J. Majer, S. Kumar, S. Girvin, and R. Schoelkopf, Nature **431**, 162 (2004).
- [9] T. Niemczyk, F. Deppe, H. Huebl, E. Menzel, F. Hocke, M. Schwarz, J. Garcia-Ripoll, D. Zueco, T. Hümmer, E. Solano, A. Marx, and R. Gross, Nature Physics **6**, 772 (2010).
- [10] I. Chiorescu, N. Groll, S. Bertaina, T. Mori, and S. Miyashita, Phys. Rev. B **82**, 024413 (2010).
- [11] D. I. Schuster, A. P. Sears, E. Ginossar, L. DiCarlo, L. Frunzio, J. J. L. Morton, H. Wu, G. A. D. Briggs, B. B. Buckley, D. D. Awschalom, and R. J. Schoelkopf, Phys. Rev. Lett. **105**, 140501 (2010).
- [12] Y. Kubo, F. R. Ong, P. Bertet, D. Vion, V. Jacques, D. Zheng, A. Dreau, J.-F. Roch, A. Auffeves, F. Jelezko, J. Wrachtrup, M.F. Barthe, P. Bergonzo, and D. Esteve, Phys. Rev. Lett. **105**, 140502 (2010).
- [13] M. Blencowe, Nature **468**, 44 (2010).
- [14] R. Amsüss, Ch. Koller, T. Nöbauer, S. Putz, S. Rotter, K. Sandner, S. Schneider, M. Schrambock, G. Steinhauser, H. Ritsch, J. Schmiedmayer, and J. Majer, Phys. Rev. Lett. **107**, 060502 (2011).
- [15] X. Zhu, S. Saito, A. Kemp, K. Kakuyanagi, S. Karimoto, H. Nakano, W. J. Munro, Y. Tokura, M. S. Everitt, K. Nemoto, M. Kasu, N. Mizuochi, and K. Semba, Nature (London) **478**, 221 (2011).
- [16] Y. Kubo, C. Grezes, A. Dewes, T. Umeda, J. Isoya, H. Sumiya, N. Morishita, H. Abe, S. Onoda, T. Ohshima, V. Jacques, A. Dreau, J.-F. Roch, I. Diniz, A. Auffeves, D. Vion, D. Esteve, and P. Bertet, **107**, 220501 (2011).
- [17] K. Hepp and E. H. Lieb, Phys. Rev. A **8**, 2517 (1973).
- [18] Y. K. Wang and F. T. Hioe, Phys. Rev. A **7**, 831 (1973).
- [19] L. A. Lugiato, in Progress in Optics, Vol. 21, pp. 71-218, edited by E. Wolf (North Holland, Amsterdam, 1984).
- [20] G. Rempe, R. J. Thompson, R. J. Brecha, W. D. Lee, and H. J. Kimble, Phys. Rev. Lett. **67**, 1727 (1991).
- [21] J. Gripp, S. L. Mielke, and L. A. Orozco, and H. J. Carmichael, Phys. Rev. A **54**, R3746 (1996).
- [22] V. M. Bastidas, C. Emary, B. Regler, T. Brandes, Phys. Rev. Lett. **108**, 043003 (2012).
- [23] B. M. Garraway, Phil. Trans. R. Soc. A **369**, 1137 (2011).
- [24] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
- [25] H. J. Carmichael, Statistical Methods in Quantum Optics 2, (2008) .
- [26] P. D. Drummond, IEEE, J. Quantum Electronics, QE-**17**, 301 (1981).
- [27] M. Murao and F. Shibata, Physica A **216**, 255, (1995).
- [28] F. Beaudoin, J. M. Gambetta, and A. Blais, Phys. Rev. A **84**, 043832 (2011).
- [29] R. Zwanzig, J. Chem. Phys. **33**, 1338 (1960).
- [30] R. Kubo, M. Toda, and N. Hashitsume, Statistical Physics II (Springer-Verlag, Berlin, 1998).
- [31] W. H. Lousell, Quantum Statistical Properties of Radiation (Wiley, New York, 1973).
- [32] L. van Hove, Physica, **21**, 517 (1955).
- [33] E. B. Davies, Quantum Theory of Open Systems (Academic, New York, 1976).
- [34] S. Tasaki, K. Yuasa, P. Facchi, G. Kimura, H. Nakazato, I. Ohba, and S. Pascazio, Annals of Physics, **322**, 631, (2007).FF
- [35] K. Saito, S. Takesue and S. Miyashita: Phys. Rev. E **61**, 2397 (2000).
- [36] K. Saito, S. Miyashita, and H. De Raedt, Phys. Rev. B **60**, 14553 (1999), K. Saito and S. Miyashita, J. Phys. Soc. Jpn. **70**, 3385 (2001)
- [37] F. Altintas and R. Eryigit, arXiv:1212.4071.
- [38] T. Mori, arXiv:1212.6726.